

# CIRCULAR STRINGS IN DE SITTER SPACETIME

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## Abstract

String propagation is investigated in de Sitter and black hole backgrounds using both exact and approximative methods. The circular string evolution in de Sitter space is discussed in detail with respect to energy and pressure, mathematical solution and physical interpretation, multi-string solutions etc. We compare with the circular string evolution in the  $2 + 1$  dimensional black hole anti de Sitter spacetime and in the equatorial plane of ordinary  $3 + 1$  dimensional stationary axially symmetric spacetime solutions of Einstein general relativity.

Using an approximative string perturbation approach we consider also generic string evolution and propagation in all these curved spacetimes.

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# 1 Introduction

The classical and quantum propagation of strings in curved spacetimes has attracted a great deal of interest in recent years. The main complication, as compared to the case of flat Minkowski space, is related to the non-linearity of the equations of motion. It makes it possible to obtain the complete analytic solution only in a very few special cases like conical spacetime [1] and plane wave/shock wave backgrounds [2]. There are however also very general results concerning integrability and solvability for maximally symmetric spacetimes [3] and gauged WZW models [4]. These are the exceptional cases, generally the string equations of motion in curved spacetimes are not integrable and even if they are, it is usually an extremely difficult task to actually separate the equations, integrate them and finally write down the complete solution in closed form. Fortunately there are different ways to "attack" a system of coupled non-linear partial differential equations. Besides numerical methods, which will not be considered here, there are essentially two ways to proceed:

**1. Approximative methods.** An approximative method was first developed by de Vega and Sánchez [5] (see also [6, 7, 8]). The idea was to expand the target space coordinates around a special solution, usually taken to be the string center of mass, and then to (try to) solve the string equations of motion and constraints order by order. This can be done, at least up to first or second order in the expansion, for most of the known gravitational and cosmological spacetimes. For a review of the method and its applications, see for instance [9].

Another approximative method, which is however not un-related to the first one, was considered in Refs.[10]. It consists of a large scale-factor expansion in spatially flat FRW-universes and an important result of the analysis was the discovery of "extremely unstable strings" of negative pressure in inflationary universes.

**2. Exact methods.** As already mentioned there are a few examples where the equations of motion and constraints can actually be solved completely. If this is not possible one can try to find special, but exact, solutions by making an ansatz. If the ansatz is properly chosen, i.e. such that it exploits the symmetries of the spacetime, it may reduce the original system of coupled non-linear partial differential equations to something much simpler, for in-

stance decoupled non-linear ordinary differential equations, and it may then be possible to find the complete solution of this reduced system. This method has been used for stationary open strings [7, 11] and for circular strings in a variety of curved spacetimes [8, 12, 13, 14, 15]. When using this method, one somehow has to argue that the special solutions under consideration are representative in the sense that their physical properties are more general than the solutions themselves. This is done by comparing with the results obtained from the approximative methods, so it is very important to have, and to use, both kinds of methods.

In this talk I will present material bases on both exact and approximative methods and I will try to take a physical point of view and address the question: what are the strings actually doing in the curved spacetimes?

The talk is organized as follows. In Section 2 we consider circular strings in stationary axially symmetric backgrounds. The main result is the effective potential determining the circular string radius. The main part of the talk concerns circular strings in de Sitter space, as indicated by the title. The material is based on Refs.[13, 14, 16] and is presented in Section 3. We first consider the energy and pressure of the different types of strings. We then turn to the mathematical solutions and their physical interpretation, following closely Ref.[13]. Finally we consider briefly the propagation of small perturbations around the circular strings. In Section 4 we consider circular strings in the background of the  $2 + 1$  dimensional black hole anti de Sitter spacetime, recently found by Bañados et. al. [17] and in Section 5 we compare with circular strings in ordinary black hole and anti de Sitter spacetimes. In Section 6 we first refine the approximative string perturbation approach of de Vega and Sánchez [5] by eliminating, from the beginning, the perturbations in the direction of the geodesic of the string center of mass. We then discuss some of the results obtained in Ref.[8], using this method, and we compare with the results obtained for circular strings.

The main conclusions of the talk are compactly summarized in tables I,II,III.

## 2 Circular Strings in Stationary Axially Symmetric Backgrounds

In this section we consider circular strings embedded in stationary axially symmetric backgrounds. The analysis is carried out in  $2 + 1$  dimensions, but the results will hold for the equatorial plane of higher dimensional backgrounds as well. To be more specific we consider the following line element:

$$ds^2 = g_{tt}(r)dt^2 + g_{rr}(r)dr^2 + 2g_{t\phi}(r)dtd\phi + g_{\phi\phi}(r)d\phi^2, \quad (2.1)$$

that will be general enough for our purposes here.

The circular string ansatz, consistent with the symmetries of the background, is taken to be:

$$t = t(\tau), \quad r = r(\tau), \quad \phi = \sigma + f(\tau), \quad (2.2)$$

where the three functions  $t(\tau)$ ,  $r(\tau)$  and  $f(\tau)$  are to be determined by the equations of motion and constraints. The equations of motion lead to:

$$\ddot{t} + 2\Gamma_{tr}^t \dot{t}\dot{r} + 2\Gamma_{\phi r}^t \dot{r}\dot{f} = 0,$$

$$\ddot{r} + \Gamma_{rr}^r \dot{r}^2 + \Gamma_{tt}^r \dot{t}^2 + \Gamma_{\phi\phi}^r (\dot{f}^2 - 1) + 2\Gamma_{t\phi}^r \dot{t}\dot{f} = 0, \quad (2.3)$$

$$\ddot{f} + 2\Gamma_{tr}^\phi \dot{t}\dot{r} + 2\Gamma_{\phi r}^\phi \dot{r}\dot{f} = 0,$$

while the constraints become:

$$g_{tt}\dot{t}^2 + g_{rr}\dot{r}^2 + g_{\phi\phi}(\dot{f}^2 + 1) + 2g_{t\phi}\dot{t}\dot{f} = 0,$$

$$g_{t\phi}\dot{t} + g_{\phi\phi}\dot{f} = 0. \quad (2.4)$$

This system of second order ordinary differential equations and constraints is most easily described as a Hamiltonian system:

$$\mathcal{H} = \frac{1}{2}g^{tt}P_t^2 + \frac{1}{2}g^{rr}P_r^2 + \frac{1}{2}g^{\phi\phi}P_\phi^2 + g^{t\phi}P_tP_\phi + \frac{1}{2}g_{\phi\phi}, \quad (2.5)$$

supplemented by the constraints:

$$\mathcal{H} = 0, \quad P_\phi = 0. \quad (2.6)$$

The function  $f(\tau)$  introduced in Eq. (2.2) does not represent a physical degree of freedom. It describes the "longitudinal" rotation of the circular string and is therefore a pure gauge artifact. This interpretation is consistent with Eq. (2.6) saying that there is no angular momentum  $P_\phi$ .

The Hamilton equations of the two cyclic coordinates  $t$  and  $f$  are:

$$\dot{f} = g^{\phi\phi} P_\phi + g^{t\phi} P_t, \quad \dot{t} = g^{tt} P_t + g^{t\phi} P_\phi, \quad (2.7)$$

as well as:

$$P_\phi = \text{const.} = 0, \quad P_t = \text{const.} \equiv -E, \quad (2.8)$$

where  $E$  is an integration constant and we used Eq. (2.6). The two functions  $t(\tau)$  and  $f(\tau)$  are then determined by:

$$\dot{f} = -E g^{t\phi}, \quad (2.9)$$

$$\dot{t} = -E g^{tt}, \quad (2.10)$$

that can be integrated provided  $r(\tau)$  is known. Using Eq. (2.6) and Eqs. (2.9)-(2.10), the Hamilton equation of  $r$  becomes after one integration:

$$\dot{r}^2 + V(r) = 0; \quad V(r) = g^{rr}(E^2 g^{tt} + g_{\phi\phi}), \quad (2.11)$$

so that  $r(\tau)$  can be obtained by inversion of:

$$\tau - \tau_o = \int_{r_o}^r \frac{dx}{\sqrt{-g^{rr}(x)[E^2 g^{tt}(x) + g_{\phi\phi}(x)]}}. \quad (2.12)$$

For the cases that we will consider in the following, Eq. (2.12) will be solved in terms of either elementary or elliptic functions. By definition of the potential  $V(r)$ , Eq. (2.11), the dynamics takes place at the  $r$ -axis in a  $(r, V(r))$ -diagram. The first thing to do for a stationary axially symmetric background under consideration, is therefore to find the zeros of the potential. Then one can try to solve the equations of motion and finally extract the physics of the problem

We close this section by the following interesting observation: insertion of the ansatz, Eq. (2.2), using the results of Eqs. (2.9)-(2.11) in the line element, Eq. (2.1), leads to:

$$ds^2 = g_{\phi\phi}(d\sigma^2 - d\tau^2). \quad (2.13)$$

We can then identify the invariant string size as:

$$S(\tau) = \sqrt{g_{\phi\phi}(r(\tau))} \quad (2.14)$$

### 3 Circular Strings in de Sitter Space

2 + 1 dimensional de Sitter space is a 3 dimensional hyperboloid embedded in 4 dimensional Minkowski space:

$$ds^2 = \frac{1}{H^2}[-(dq^0)^2 + \sum_{i=1}^3 (dq^i)^2], \quad \eta_{\mu\nu} q^\mu q^\nu = 1. \quad (3.1)$$

The hyperboloid coordinates cover the whole manifold contrary to the co-moving coordinates:

$$ds^2 = -(dX^0)^2 + e^{2HX^0}[dR^2 + R^2 d\phi^2], \quad (3.2)$$

and the static coordinates:

$$ds^2 = -(1 - H^2 r^2) dt^2 + \frac{dr^2}{1 - H^2 r^2} + r^2 d\phi^2. \quad (3.3)$$

For our purposes it is however most convenient to start out with the static coordinates since then we can use directly the results of Section 2 for the circular strings. Afterwards, we then have to transform the solutions back to the hyperboloid using the appropriate coordinate transformations inside and outside the horizon.

In static coordinates we can now immediately write down the potential, Eq. (2.11), determining the dynamics of circular strings in de Sitter space:

$$V(r) = r^2 - H^2 r^4 - E^2, \quad (3.4)$$

see Fig.1. For  $4H^2 E^2 < 1$  the top of the potential is above the  $r$ -axis and there will be oscillating solutions to the left of the barrier and bouncing

solutions to the right. The bouncing solutions will re-expand towards infinity. For  $4H^2E^2 > 1$ , on the other hand, the top of the potential is below the  $r$ -axis and therefore the potential does not act as a barrier. Strings can expand from  $r = 0$  towards infinity and very large strings can collapse. Finally if  $4H^2E^2 = 1$  there is a (unstable) stationary string at the top of the potential as well as expanding and collapsing strings. It is instructive to compare with the circular string dynamics in Minkowski space. In that case the potential is given by  $V(r) = r^2 - E^2$  and only oscillating strings exist. Physically the difference is not so difficult to understand. In flat Minkowski space the dynamics of a circular string is determined by the string tension only, that will always try to contract the string. This leads to strings oscillating between  $r = 0$  and some maximal size depending on the energy. In de Sitter space there is an opposite effect namely the expansion of the universe, that will try to expand the string. This gives rise to a region where the tension is strongest (to the left of the barrier), a region where the expansion of the universe is strongest (to the right of the barrier) and there may even be an exact balance of the two opposite forces, implying the existence of a stationary solution.

Before coming to the mathematical solution of the equations of motion, we consider the energy and pressure of the strings. The spacetime string energy-momentum tensor is:

$$\sqrt{-g}T^{\mu\nu}(X) = \frac{1}{2\pi\alpha'} \int d\sigma d\tau (\dot{X}^\mu \dot{X}^\nu - X'^\mu X'^\nu) \delta^{(3)}(X - X(\tau, \sigma)). \quad (3.5)$$

After integration over a spatial volume that completely encloses the string, the energy-momentum tensor takes the form of a fluid:

$$T^\mu_\nu = \text{diag.}(-En., Pr., Pr.), \quad (3.6)$$

where, in the comoving coordinates introduced in Eq. (3.2):

$$En.(X) = \frac{1}{\alpha'} \dot{X}^0, \quad (3.7)$$

$$Pr.(X) = \frac{1}{2\alpha'} \frac{e^{2HX^0}(\dot{R}^2 - R^2)}{\dot{X}^0}, \quad (3.8)$$

represent the energy and pressure, respectively. From these expressions we can actually get a lot of information about the energy and pressure, without using the explicit time evolution of the strings. From the coordinate

transformations between comoving and static coordinates we get the explicit expressions [16]:

$$H\alpha'En. = \frac{H^2r\dot{r} - HE}{H^2r^2 - 1}, \quad (3.9)$$

$$H\alpha'Pr. = \frac{(H^4r^4 - 2H^2r^2 - H^2E^2)H^2r\dot{r} + (3H^4r^4 - 2H^2r^2 + H^2E^2)HE}{2(1 - H^2r^2)(H^4r^4 + H^2E^2)}. \quad (3.10)$$

Both the energy and the pressure now depend on the string radius  $r$  and the velocity  $\dot{r}$ . The latter can however be eliminated using Eq. (2.11) and Eq. (3.4).

Let us first consider a string expanding from  $r = 0$  towards infinity. This corresponds to a string with  $\dot{r} > 0$  and  $H^2E^2 > 1/4$ , see Fig.1. For  $r = 0$  we find  $En. = E/\alpha'$  and  $Pr. = E/(2\alpha')$ , thus the equation of state is  $Pr. = En./2$ . This is like ultra-relativistic matter. As the string expands, the energy soon starts to increase while the pressure starts to decrease and becomes negative, see Fig.2. For  $r \rightarrow \infty$  we find  $En. = r/\alpha'$  and  $Pr. = -r/(2\alpha')$ , thus not surprisingly, we have recovered the equation of state of extremely unstable strings  $Pr. = -En./2$ . [10]. Now consider an oscillating string, i.e. a string with  $H^2E^2 \leq 1/4$  in the region to the left of the potential barrier, Fig.1. The equation of state near  $r = 0$  is the same as for the expanding string, but now the string has a maximal radius:

$$Hr_{\max} \equiv \nu = \sqrt{\frac{1 - \sqrt{1 - 4H^2E^2}}{2}}. \quad (3.11)$$

For  $r = r_{\max}$  we find:

$$H\alpha'En. = \frac{\nu}{\sqrt{1 - \nu^2}} \equiv k, \quad H\alpha'Pr. = \frac{k}{2}(-1 + \frac{2k^2}{1 + k^2}), \quad (3.12)$$

corresponding to a perfect fluid type equation of state:

$$Pr. = (\gamma - 1)En., \quad \gamma = \frac{k^2}{1 + k^2} + \frac{1}{2}. \quad (3.13)$$

Notice that  $k \in [0, 1]$ , with  $k = 0$  describing a string at the bottom of the potential while  $k = 1$  describes a string oscillating between  $r = 0$  and the top of the potential barrier (in this case the string actually only makes one



oscillation [12, 13]). For  $k \ll 1$  the equation of state, Eq.(3.13), near the maximal radius reduces to  $Pr. = -En./2$ . In the other limit  $k \rightarrow 1$  we find, however,  $Pr. \rightarrow 0$  corresponding to cold matter. For the oscillating strings we can also calculate the average values of energy and pressure by integrating over a full period, see Fig.3. Since a  $\tau$ -integral can be converted into a  $r$ -integral, using Eq. (2.11) and Eq. (3.4), the average values can be obtained without using the exact  $\tau$ -dependence of the string radius. The average energy becomes:

$$H\alpha' < En. > = \frac{2k}{T\sqrt{1+k^2}} \Pi\left(\frac{-k^2}{1+k^2}, k\right), \quad (3.14)$$

where  $T$  is the period and  $\Pi$  is the complete elliptic integral of the third kind (using the notation of Gradshteyn [18]). The average pressure is zero, thus in average the oscillating strings describe cold matter. In Minkowski space, for comparison, it can easily be shown [16] that the energy is constant while the pressure depends on the string radius. The average pressure is however zero, just as in de Sitter space.

### 3.1 The Mathematical Solution

(This and the following subsection follow closely Ref.[13].)

We now come to the explicit mathematical solutions. In the general case equations (2.11) and (3.4) are solved by:

$$H^2 r^2(\tau) = \wp(\tau - \tau_o) + 1/3, \quad (3.15)$$

where  $\wp$  is the Weierstrass elliptic  $\wp$ -function [19]:

$$\dot{\wp}^2 = 4\wp^3 - g_2\wp - g_3, \quad (3.16)$$

with invariants:

$$g_2 = 4\left(\frac{1}{3} - H^2 E^2\right), \quad g_3 = \frac{4}{3}\left(\frac{2}{9} - H^2 E^2\right), \quad (3.17)$$

and discriminant:

$$\Delta \equiv g_2^3 - 27g_3^2 = 16H^4 E^4 (1 - 4H^2 E^2). \quad (3.18)$$

It is convenient to consider separately the 3 cases  $H^2 E^2 = 1/4$  ( $\Delta = 0$ ),  $H^2 E^2 < 1/4$  ( $\Delta > 0$ ) and  $H^2 E^2 > 1/4$  ( $\Delta < 0$ ), see Fig.1.

**$H^2 E^2 = 1/4$ :** In this case the Weierstrass function reduces to a hyperbolic function and Eq. (3.15) becomes:

$$H^2 r^2(\tau) = \frac{1}{2} \left( 1 + \sinh^{-2} \left( \frac{\tau - \tau_o}{\sqrt{2}} \right) \right). \quad (3.19)$$

Two real independent solutions are obtained by the choices  $\tau_o = 0$  and  $\tau_o = i\pi/\sqrt{2}$ , respectively:

$$H^2 r_-^2(\tau) = \frac{1}{2} \tanh^{-2} \frac{\tau}{\sqrt{2}}, \quad (3.20)$$

$$H^2 r_+^2(\tau) = \frac{1}{2} \tanh^2 \frac{\tau}{\sqrt{2}}. \quad (3.21)$$

Notice that:

$$\begin{aligned} H^2 r_-^2(-\infty) &= \frac{1}{2}, & H^2 r_-^2(0) &= \infty, & H^2 r_-^2(\infty) &= \frac{1}{2}, \\ H^2 r_+^2(-\infty) &= \frac{1}{2}, & H^2 r_+^2(0) &= 0, & H^2 r_+^2(\infty) &= \frac{1}{2}. \end{aligned} \quad (3.22)$$

These are the 2 solutions originally found by de Vega, Sánchez and Mikhailov [12], corresponding to  $\alpha > 0$  and  $\alpha < 0$  in their notation, respectively. The interpretation of these solutions as a function of the world-sheet time  $\tau$  is clear from Fig.1: the solution, Eq. (3.20), expands from  $H^2 r_-^2 = 1/2$  towards infinity and then contracts until it reaches its original size. The solution, Eq. (3.21), contracts from  $H^2 r_+^2 = 1/2$  until it collapses. It then expands again until it reaches its original size. The physical interpretation, that is somewhat more involved, was described in Ref.[12] and will be shortly reviewed in Subsection 3.2. There is actually also a stationary solution for  $H^2 E^2 = 1/4$ , i.e. a string sitting on the top of the potential, see Fig.1. This solution with constant string size  $S = 1/(\sqrt{2}H)$  was discussed in Ref.[12] and will not be considered here.

**$H^2 E^2 < 1/4$ :** Here two real independent solutions are obtained by the choices  $\tau_o = 0$  and  $\tau_o = \omega'$ , respectively:

$$H^2 r_-^2(\tau) = \wp(\tau) + 1/3, \quad (3.23)$$

$$H^2 r_+^2(\tau) = \wp(\tau + \omega') + 1/3, \quad (3.24)$$

where  $\omega'$  is the imaginary semi-period of the Weierstrass function. It is explicitly given by [19]:

$$\omega' = i \frac{\sqrt{2} K'(k)}{\sqrt{1 + \sqrt{1 - 4H^2 E^2}}}; \quad k = \sqrt{\frac{1 - \sqrt{1 - 4H^2 E^2}}{1 + \sqrt{1 - 4H^2 E^2}}}. \quad (3.25)$$

Notice that:

$$\begin{aligned} H^2 r_-^2(0) &= \infty, & H^2 r_-^2(\omega) &= (1 + \sqrt{1 - 4H^2 E^2})/2, & H^2 r_-^2(2\omega) &= \infty, \dots \\ H^2 r_+^2(0) &= 0, & H^2 r_+^2(\omega) &= (1 - \sqrt{1 - 4H^2 E^2})/2, & H^2 r_+^2(2\omega) &= 0, \dots \end{aligned} \quad (3.26)$$

where  $\omega$  is the real semi-period of the Weierstrass function [19]:

$$\omega = \frac{\sqrt{2} K(k)}{\sqrt{1 + \sqrt{1 - 4H^2 E^2}}}, \quad (3.27)$$

and  $K'$  and  $K$  are the complete elliptic integrals of first kind. The interpretation of these solutions as a function of  $\tau$  is clear from Fig.1: the solution, Eq. (3.23), oscillates between infinity and its minimal size  $H^2 r_-^2 = (1 + \sqrt{1 - 4H^2 E^2})/2$  at the boundary of the potential, while the solution, Eq. (3.24), oscillates between 0 and its maximal size  $H^2 r_+^2 = (1 - \sqrt{1 - 4H^2 E^2})/2$ . The physical interpretation will be considered in Subsection 3.2.

**$H^2 E^2 > 1/4$ :** In this last case two real independent solutions are obtained by the choices  $\tau_o = 0$  and  $\tau_o = \omega'_2$ , respectively:

$$H^2 r_-^2(\tau) = \wp(\tau) + 1/3, \quad (3.28)$$

$$H^2 r_+^2(\tau) = \wp(\tau + \omega'_2) + 1/3, \quad (3.29)$$

where  $\omega'_2$  takes the explicit form:

$$\omega'_2 = i \frac{K'(\hat{k})}{\sqrt{HE}}; \quad \hat{k} = \sqrt{\frac{1}{2} + \frac{1}{4HE}}. \quad (3.30)$$

Notice that:

$$\begin{aligned} H^2 r_-^2(0) &= \infty, & H^2 r_-^2(\omega_2) &= 0, & H^2 r_-^2(2\omega_2) &= \infty, \dots \\ H^2 r_+^2(0) &= 0, & H^2 r_+^2(\omega_2) &= \infty, & H^2 r_+^2(2\omega_2) &= 0, \dots \end{aligned} \quad (3.31)$$

where:

$$\omega_2 = \frac{K(\hat{k})}{\sqrt{HE}}. \quad (3.32)$$

It should be stressed that in this case the primitive semi-periods are  $\hat{\omega} = (\omega_2 - \omega'_2)/2$  and  $\hat{\omega}' = (\omega_2 + \omega'_2)/2$ , i.e.  $(2\hat{\omega}, 2\hat{\omega}')$  spans a fundamental period parallelogram in the complex plane.

The interpretation of the solutions, Eqs. (3.28)-(3.29), as a function of  $\tau$  follows from Fig.1: both of them oscillates between zero size (collapse) and infinite size (instability). The physical interpretations will follow in Subsection 3.2.

### 3.2 The Physical Interpretation

$H^2E^2 = 1/4$ : We first consider the  $r_-$ -solution, Eq. (3.20). The hyperboloid time is obtained by integrating Eq. (2.10) and transforming back to the hyperboloid coordinates:

$$q_-^0(\tau) = \sinh \tau - \frac{1}{\sqrt{2}} \cosh \tau \coth \frac{\tau}{\sqrt{2}}. \quad (3.33)$$

When we plot this function (Fig.4.) we see that the string solution actually describes 2 strings (I and II) [12], since  $\tau$  is a two-valued function of  $q_-^0$ . For both strings the invariant string size is:

$$S_- = \frac{1}{\sqrt{2}H} \coth \left| \frac{\tau}{\sqrt{2}} \right|, \quad (3.34)$$

but string I corresponds to  $\tau \in ]-\infty, 0[$  and string II to  $\tau \in ]0, \infty[$ . Therefore,  $q_-^0 \rightarrow \infty$  corresponds to  $\tau \rightarrow 0_-$  for string I, but to  $\tau \rightarrow \infty$  for string II. More generally, when  $q_-^0 \rightarrow \infty$  the invariant size grows indefinitely for string I, while it approaches a constant value for string II. We conclude that string I is an unstable string for  $q_-^0 \rightarrow \infty$ , while string II is a stable string. More details about the connection between hyperboloid time and world-sheet time for these solutions can be found in Ref.[12]. Let us consider now the comoving time of the solution  $r_-$  in a little more detail:

$$X_-^0(\tau) = \frac{1}{H}(\tau + \log \left| \frac{1}{\sqrt{2}} \coth \frac{\tau}{\sqrt{2}} - 1 \right|). \quad (3.35)$$

When we plot this function (Fig.4.) we find that  $\tau$  is a three-valued function of  $X_-^0$ . What happens is that the time interval  $\tau \in ]0, \infty[$  for string II splits into two parts. These features are easily understood when returning to the effective potential, Fig.1.: String I starts at  $H^2 r_-^2 = 1/2$  for  $\tau = HX_-^0 = -\infty$ , it then expands through the horizon  $H^2 r_-^2 = 1$  at:

$$\tau = -\sqrt{2} \log(1 + \sqrt{2}), \quad HX_-^0 = \log 2 - \sqrt{2} \log(1 + \sqrt{2}) \quad (3.36)$$

and continues towards infinity for  $\tau \rightarrow 0_-$ ,  $HX_-^0 \rightarrow \infty$ . String II starts at infinity for  $\tau = 0_+$ ,  $HX_-^0 = \infty$  and contracts through the horizon at:

$$\tau = \sqrt{2} \log(1 + \sqrt{2}), \quad HX_-^0 = -\infty. \quad (3.37)$$

This behaviour, approaching the horizon from the outside, corresponds to the going backwards in comoving time-part of Fig.4. String II then continues contracting from  $H^2 r_-^2 = 1$  at:

$$\tau = \sqrt{2} \log(1 + \sqrt{2}), \quad HX_-^0 = -\infty, \quad (3.38)$$

until it reaches  $H^2 r_-^2 = 1/2$  at  $\tau = HX_-^0 = \infty$ . We now consider briefly the  $r_+$ -solution, Eq. (3.21). In this case the hyperboloid time is given by:

$$q_+^0(\tau) = \sinh \tau - \frac{1}{\sqrt{2}} \cosh \tau \tanh \frac{\tau}{\sqrt{2}}, \quad (3.39)$$

which is a monotonically increasing function of  $\tau$ . The  $r_+$ -solution therefore describes only one string. The proper size is given by:

$$S_+ = \frac{1}{\sqrt{2}H} \tanh \left| \frac{\tau}{\sqrt{2}} \right| \quad (3.40)$$

We can also express this solution in terms of the comoving time:

$$X_+^0(\tau) = \frac{1}{H} \left( \tau + \log \left( 1 - \frac{1}{\sqrt{2}} \tanh \frac{\tau}{\sqrt{2}} \right) \right), \quad (3.41)$$

but since everything now takes place well inside the horizon, this will not really give us more insight. The string starts with  $H^2 r_+^2 = 1/2$  for  $\tau = HX_+^0 = -\infty$ . It then contracts until it collapses for  $\tau = HX_+^0 = 0$  and expands again and eventually reaches  $H^2 r_+^2 = 1/2$  for  $\tau = HX_+^0 = \infty$ .

$\mathbf{H}^2\mathbf{E}^2 < 1/4$ : Using the notation introduced in Eqs. (3.11)-(3.12) the solutions, Eqs. (3.23)-(3.24), can be written as:

$$H^2 r_-^2(\tau) = \frac{\mu^2}{\text{sn}^2[\mu\tau \mid k]}, \quad (3.42)$$

$$H^2 r_+^2(\tau) = \nu^2 \text{sn}^2[\mu\tau \mid k], \quad (3.43)$$

where  $\mu = \sqrt{1 - \nu^2}$ . Consider first the  $r_-$ -solution, Eq. (3.42). It is clear from Eq. (3.26) and the periodicity in general that we have infinitely many branches  $[0, 2\omega]$ ,  $[2\omega, 4\omega]$ , .... We will see in a moment that each of these branches actually corresponds to one string, that is, the  $r_-$ -solution describes infinitely many strings. For that purpose we will need the hyperboloid time and the comoving time as a function of  $\tau$ . Both of them are expressed in terms of the static coordinate time  $t$ , that is obtained by integrating Eq. (2.10):

$$Ht_-(\tau) = \zeta(x/\mu)\tau + \frac{1}{2} \log \left| \frac{\sigma(\tau - x/\mu)}{\sigma(\tau + x/\mu)} \right|, \quad (3.44)$$

where  $\zeta$  and  $\sigma$  are the Weierstrass  $\zeta$  and  $\sigma$ -functions [19] and  $x$  is a real constant obeying  $\text{sn}[x \mid k] = \mu$ , i.e.  $x$  is expressed as an incomplete elliptic integral of the first kind. The expression, Eq.(3.44), can be further rewritten in terms of theta-functions [19]:

$$Ht_-(\tau) = \frac{\mu\tau\pi}{2K} \frac{\vartheta_1'}{\vartheta_1}\left(\frac{\pi x}{2K}\right) + \frac{1}{2} \log \left| \frac{\vartheta_1\left(\frac{\pi(\mu\tau-x)}{2K}\right)}{\vartheta_1\left(\frac{\pi(\mu\tau+x)}{2K}\right)} \right|, \quad (3.45)$$

and finally as:

$$Ht_-(\tau) = \frac{1}{2} \log \left| \frac{\sin\left(\frac{\pi(\mu\tau-x)}{2K}\right)}{\sin\left(\frac{\pi(\mu\tau+x)}{2K}\right)} \right| + \frac{\mu\tau\pi}{2K} \frac{\vartheta_1'}{\vartheta_1}\left(\frac{\pi x}{2K}\right) - 2 \sum_{n=1}^{\infty} \frac{q^{2n}}{n(1-q^{2n})} \sin\left(\frac{n\pi\mu\tau}{K}\right) \sin\left(\frac{n\pi x}{K}\right), \quad (3.46)$$

where  $q = e^{-\pi K'/K}$ . In the latter expression we have isolated all the real singularities in the first term. To be more specific we see that the static coordinate time is singular for  $\mu\tau \rightarrow 2KN \pm x$ , where  $N$  is an integer, with the asymptotic behaviour:

$$t_-(\tau) \rightarrow \pm \frac{1}{2H} \log \left| \mu\tau - 2KN \mp x \right|; \quad \tau \rightarrow \frac{2K}{\mu} N \pm \frac{x}{\mu}. \quad (3.47)$$

On the other hand  $t_-(\tau)$  is completely regular at the boundaries of the branches, i.e. for  $\tau = 0, \pm 2\omega, \pm 4\omega, \dots$ . These results can be easily translated to the hyperboloid time [13]:

$$q_-^0(\tau) = -\frac{\Omega\vartheta_1'(0)}{2\pi} \frac{e^{\Omega\tau\frac{\vartheta_1'}{\vartheta_1}(\Omega y)}\vartheta_1(\Omega(y-\tau)) + e^{-\Omega\tau\frac{\vartheta_1'}{\vartheta_1}(\Omega y)}\vartheta_1(\Omega(y+\tau))}{\vartheta_1(\Omega\tau)\vartheta_1(\Omega y)}, \quad (3.48)$$

where:

$$\Omega \equiv \frac{\pi\mu}{2K}, \quad y \equiv \frac{x}{\mu}. \quad (3.49)$$

Notice that the singularities, Eq. (3.47), that originated from the zeros of  $\vartheta_1(\Omega(y \pm \tau))$ , have canceled in  $q_-^0(\tau)$  so that  $q_-^0(2\omega N \pm y)$  is finite.  $q_-^0(\tau)$  blows up for  $\tau = 0, \pm 2\omega, \pm 4\omega, \dots$ , like:

$$|q_-^0| \propto \left| \frac{1}{2N\omega - \tau} \right|, \quad (3.50)$$

where  $N$  is again an integer. This demonstrates that the world-sheet time  $\tau$  is actually an infinite valued function of  $q_-^0$ , and that the solution  $r_-$  therefore describes *infinitely many strings* (see Fig.5). This should be compared with the  $H^2E^2 = 1/4$  case where we found a solution describing two strings. In that case the two strings were of completely different type and had completely different physical interpretations. In the present case we find infinitely many strings but they are all of the same type. In the branch  $\tau \in [0, 2\omega]$  (say) the string starts with infinite string size at  $\tau = 0$ ,  $q_-^0 = -\infty$ . It then contracts to its minimal size  $H^2r_-^2 = (1 + \sqrt{1 - 4H^2E^2})/2$  and reexpands towards infinity at  $\tau = 2\omega$ ,  $q_-^0 = \infty$ . This solution, and the infinitely many others of the same type, are unstable strings.

The comoving time of the  $r_-$ -solution is given by [13]:

$$\begin{aligned} HX_-^0(\tau) &= \Omega\tau\frac{\vartheta_1'}{\vartheta_1}(\Omega y) + \log \left| \frac{\Omega\vartheta_1(\Omega(\tau-y))\vartheta_1'(0)}{\pi\vartheta_1(\Omega\tau)\vartheta_1(\Omega y)} \right| \\ &= \log \left| \frac{\Omega \sin(\Omega(\tau-y))}{\sin(\Omega\tau)} \right| - \log \left| \frac{\pi\vartheta_1(\Omega y)}{\vartheta_1'(0)} \right| + \Omega\tau\frac{\vartheta_1'}{\vartheta_1}(\Omega y) \\ &\quad - 4 \sum_{m=1}^{\infty} \frac{q^{2m}}{m(1-q^{2m})} \sin(m\Omega y) \sin(m\Omega(2\tau - y)). \end{aligned} \quad (3.51)$$

It can be shown that  $0 \leq \Omega \leq 1$  and  $1.246.. \leq y \leq \pi/2$ . It follows that:

$$0 < y < \omega < 2\omega - y < 2\omega. \quad (3.52)$$

The comoving time, Eq. (3.51), is singular at  $\tau = 0, y, 2\omega$  but regular at  $\tau = 2\omega - y$  and similarly in the other branches, Fig.5. Therefore, the interpretation of the string solution in the branch  $\tau \in [0, 2\omega]$  (say), as seen in comoving coordinates, is as follows: The string starts with infinite size at  $\tau = 0$ ,  $HX_-^0 = \infty$ . It then contracts and passes the horizon from the outside at  $\tau = y$ ,  $HX_-^0 = -\infty$ . The string now continues contracting from the inside of the horizon at  $\tau = y$ ,  $HX_-^0 = -\infty$  until it reaches the minimal size at:

$$\tau = \omega, \quad HX_-^0 = \frac{\pi}{2} \frac{\vartheta_1'}{\vartheta_1}(\Omega y) + \frac{1}{2} \log \frac{1 - \sqrt{1 - 4H^2 E^2}}{2}. \quad (3.53)$$

From now on the string expands again. It passes the horizon from the inside after finite comoving time and continues towards infinity for  $HX_-^0 \rightarrow \infty$ .

It is an interesting observation that the comoving time is not periodic in  $\tau$ , i.e.  $X_-^0(\tau) \neq X_-^0(\tau + 2\omega)$ , although the string size is. Explicitly we find:

$$X_-^0(\tau + 2\omega) - X_-^0(\tau) = \frac{\pi}{H} \frac{\vartheta_1'}{\vartheta_1}(\Omega y). \quad (3.54)$$

This means that  $r_-(\tau)$  really describes infinitely many strings with different invariant size at a given comoving time. To be more specific let us consider a fixed comoving time  $X_-^0$  and the corresponding world-sheet times:

$$X_-^0 \equiv X_-^0(\tau_1) = X_-^0(\tau_2) = \dots, \quad (3.55)$$

where  $\tau_1 \in [0, 2\omega[$ ,  $\tau_2 \in [2\omega, 4\omega[$ .... Taking for simplicity a comoving time  $HX_-^0 \gg 1$  we have (see Fig.5.):

$$\tau_n = \frac{n\pi}{\Omega} + \epsilon_n, \quad \epsilon_n \ll 1. \quad (3.56)$$

To the lowest orders we find from Eq. (3.51):

$$HX_-^0 = -\log \epsilon_n + n\pi \frac{\vartheta_1'}{\vartheta_1}(\Omega y) + \mathcal{O}(\epsilon_n), \quad (3.57)$$



so that:

$$\epsilon_n = \exp[-HX_-^0 + n\pi \frac{\vartheta'_1}{\vartheta_1}(\Omega y) + \dots] \quad (3.58)$$

The invariant string sizes are then:

$$HS_- \approx \frac{1}{\epsilon_n} = \exp[HX_-^0 - n\pi \frac{\vartheta'_1}{\vartheta_1}(\Omega y) + \dots] \quad (3.59)$$

i.e. they are separated by a multiplicative factor. This expression, of course, is only valid as long as  $\epsilon_n \ll 1$ , so  $n$  should not be too large.

We now consider the  $r_+$ -solution, Eq. (3.43). In this case the dynamics takes place well inside the horizon. The possible singularities of the hyperboloid time  $q_+^0$  and the comoving time  $X_+^0$  therefore coincide with the singularities of the static coordinate time  $t_+$ . The static coordinate time is again obtained from Eq. (2.10), which we first rewrite as:

$$H\dot{t}_+(\tau) = \frac{HE}{2/3 - \wp(\tau + \omega')} = HE \left(1 + \frac{H^2 E^2}{\wp(\tau) - (H^2 E^2 - 1/3)}\right). \quad (3.60)$$

Integration leads to:

$$Ht_+(\tau) = \tau(HE + \zeta(a)) + \frac{1}{2} \log \left| \frac{\sigma(\tau - a)}{\sigma(\tau + a)} \right|, \quad (3.61)$$

where  $a$  is a complex constant obeying  $\wp(a) = H^2 E^2 - 1/3$ , i.e.  $\text{sn}[a\mu \mid k] = 1/\nu$ . It follows that  $a\mu = iK + x$  where  $x$  is real and  $\text{sn}[x \mid k] = \mu$ . Again we can express the static coordinate time in terms of theta-functions [19]:

$$Ht_+(\tau) = \tau(HE + \frac{\mu\pi}{2K} \frac{\vartheta'_4}{\vartheta_4}(\frac{\pi x}{2K})) + \frac{1}{2} \log \left| \frac{\vartheta_4(\frac{\pi(\mu\tau - x)}{2K})}{\vartheta_4(\frac{\pi(\mu\tau + x)}{2K})} \right|, \quad (3.62)$$

or in terms of the Jacobi zeta-function  $zn$  [18]:

$$Ht_+(\tau) = \tau(HE + \mu zn(x, k)) - 2 \sum_{n=1}^{\infty} \frac{q^n}{n(1 - q^{2n})} \sin\left(\frac{n\pi\mu\tau}{K}\right) \sin\left(\frac{n\pi x}{K}\right), \quad (3.63)$$

where  $q = e^{-\pi K'/K}$ . In this form we see explicitly that  $t_+$  consists of a linear term plus oscillating terms. The comoving time takes the form:

$$HX_+^0(\tau) = \frac{1}{2} \log(1 - \nu^2 \text{sn}^2[\mu\tau \mid k]) + Ht_+(\tau)$$

$$= \tau[HE + \Omega \frac{\vartheta'_4}{\vartheta_4}(\Omega y)] + \log \left| \frac{\Omega \vartheta_4(\Omega(\tau-y)) \vartheta'_1(0)}{\pi \vartheta_1(\Omega y) \vartheta_4(\Omega \tau)} \right|. \quad (3.64)$$

Notice that the argument of the log has no real zeros:

$$HX_+^0(\tau) = \tau[HE + \Omega \frac{\vartheta'_4}{\vartheta_4}(\Omega y)] + \log \left| \frac{\Omega \vartheta'_1(0)}{\pi \vartheta_1(\Omega y)} \right| - 4 \sum_{n=1}^{\infty} \frac{q^n}{n(1-q^{2n})} \sin(n\pi\mu\Omega) \sin(n\pi\Omega(2\tau - y)). \quad (3.65)$$

The static coordinate time and the cosmic time are therefore completely regular functions of  $\tau$ , and it follows that the string solution  $r_+$ , which is *oscillating regularly* as a function of world-sheet time  $\tau$ , is also oscillating regularly when expressed in terms of hyperboloid time or comoving time. This solution represents one *stable* string.

**$H^2 E^2 > 1/4$ :** The analysis here is very similar to the analysis of the  $r_-$ —solution in the  $H^2 E^2 < 1/4$ —case so we shall not go into it here. The results are summarized in Table I, together with the results from the other cases.

**Table I**

Circular string evolution in de Sitter spacetime. For each  $H^2 E^2$ , there exists two independent solutions  $r_-$  and  $r_+$ :

$H^2 E^2$	$r_-$	$r_+$
$< 1/4$	Infinitely many different strings. All are unstable ( $ Hr_-^{max} = \infty$ ), and never collapse. ( $ Hr_-^{min} = \sqrt{(1 + \sqrt{1 - 4H^2 E^2})/2} > 0$ ).	One stable oscillating string ( $ 0 \leq r_+ \leq r_+^{max} ; Hr_+^{max} = \sqrt{(1 - \sqrt{1 - 4H^2 E^2})/2}$ ).
$> 1/4$	Infinitely many strings. All of them are unstable ( $ Hr_-^{max} = \infty$ ) and they collapse to a point ( $ r_-^{min} = 0$ ).	Infinitely many strings similar to $r_-$ . In this case $r_+$ is just a time-translation of $r_-$ : $ r_+(\tau) = r_-(\tau + \omega_2)$ .
$= 1/4$	Two different and non-oscillating strings $ r_-^{(I)}$ and $ r_-^{(II)}$ . $ r_-^{(I)}$ is unstable and $ r_-^{(II)}$ is stable for large de Sitter radius.	One stable and non-oscillating string (it makes only one oscillation). $ Hr_+^{min} = 0$ , $ Hr_+^{max} = 1/\sqrt{2}$ .

### 3.3 Small Perturbations Around Circular Strings

Using the covariant approach of Frolov and Larsen [7], we have considered small perturbations propagating along the circular strings in black hole and cosmological spacetime backgrounds [14, 20]. In the case of the 3 + 1 dimensional de Sitter space introduce two normal vectors  $n_\perp^\mu$  and  $n_\parallel^\mu$  perpendicular

to the string world-sheet:

$$n_{\perp}^{\mu} = (0, 0, \frac{1}{r}, 0), \quad n_{\parallel}^{\mu} = (\frac{\dot{r}}{r(1-H^2r^2)}, \frac{E}{r}, 0, 0), \quad (3.66)$$

and then consider only physical perturbations:

$$\delta x^{\mu} = \delta x^{\perp} n_{\perp}^{\mu} + \delta x^{\parallel} n_{\parallel}^{\mu}, \quad (3.67)$$

where  $\delta x^{\perp}$  and  $\delta x^{\parallel}$  are the perturbations as seen by an observer travelling with the circular string. After Fourier expanding  $\delta x^R$ :

$$\delta x^R(\tau, \sigma) = \sum_n C_n^R(\tau) e^{-in\sigma}, \quad R = \perp, \parallel \quad (3.68)$$

it can be shown that [14]:

$$\ddot{C}_{n\perp} + (n^2 - 2H^2r^2)C_{n\perp} = 0, \quad (3.69)$$

$$\ddot{C}_{n\parallel} + (n^2 - 2H^2r^2 - \frac{2E^2}{r^2})C_{n\parallel} = 0, \quad (3.70)$$

determining the comoving perturbations. These equations have been discussed in detail in Refs.[14, 20], so we shall just give one simple result here. For  $r \rightarrow \infty$  the brackets in Eqs. (3.69)-(3.70) become negative. This means that the perturbations develop imaginary frequencies and grow indefinitely. However, by considering the detailed solutions it turns out that the perturbations grow with the same rate as the radius of the underlying circular string (which by the way grows with the same rate as the universe) so although the perturbations grow, the circular shape of the string is actually stable. More details can be found in Refs.[14, 20].

## 4 Circular Strings in the 2+1 BH-ADS Space-time

We now consider the circular string dynamics in the 2+1 black hole anti de Sitter (BH-ADS) spacetime recently found by Bañados et. al. [17]. This

spacetime background has arised much interest recently. It describes a two-parameter family (mass  $M$  and angular momentum  $J$ ) of black holes in 2+1 dimensional general relativity with metric:

$$ds^2 = (M - \frac{r^2}{l^2})dt^2 + (\frac{r^2}{l^2} - M + \frac{J^2}{4r^2})^{-1}dr^2 - Jdtd\phi + r^2d\phi^2. \quad (4.1)$$

It has two horizons  $r_{\pm} = \sqrt{\frac{Ml^2}{2} \pm \frac{l}{2}\sqrt{M^2l^2 - J^2}}$  and a static limit  $r_{\text{erg}} = \sqrt{M}l$ , defining an ergosphere, as for ordinary Kerr black holes. Using the general formalism of Section 2, we can immediately read off the potential (see Fig.6.):

$$V(r) = r^2(\frac{r^2}{l^2} - M) + \frac{J^2}{4} - E^2. \quad (4.2)$$

The potential, Eq. (4.2), has a global minimum between the two horizons:

$$V_{\min} = V(\sqrt{\frac{Ml^2}{2}}) = -\frac{1}{4}(M^2l^2 - J^2 + 4E^2) < 0, \quad (4.3)$$

which is always negative, since we only consider the case when  $Ml^2 \geq J^2$  (otherwise there are no horizons). For large values of  $r$  the potential goes as  $r^4$  and at  $r = 0$  we have:

$$V(0) = \frac{J^2}{4} - E^2, \quad (4.4)$$

that can be either positive, negative or zero. Notice also that the potential vanishes provided:

$$V(r_0) = 0 \Leftrightarrow r_{01,2} = \sqrt{\frac{Ml^2}{2} \pm \frac{l}{2}\sqrt{M^2l^2 - J^2 + 4E^2}}. \quad (4.5)$$

There are therefore three fundamentally different types of solutions.

**(i):** For  $J^2 > 4E^2$  there are two positive- $r$  zeros of the potential (Fig.6a). The smallest zero is located between the inner horizon and  $r = 0$ , while the other zero is between the outer horizon and the static limit. Therefore, this string solution never comes outside the static limit. On the other hand it never falls into  $r = 0$ . The mathematical solution oscillating between these two positive zeros of the potential may be interpreted as a string travelling between the different universes described by the maximal analytic extension

of the spacetime (the Penrose diagram of the  $2 + 1$  dimensional BH-ADS spacetime is discussed in Refs.[21, 22]). Such type of circular string solutions also exist in other stringy black hole backgrounds [23].

**(ii):** For  $J^2 < 4E^2$  there is only one positive- $r$  zero of the potential, which is always located outside the static limit (Fig.6b). The potential is negative for  $r = 0$ , so there is no barrier preventing the string from collapsing into  $r = 0$ . By suitably fixing the initial conditions the string starts with its maximal size outside the static limit at  $\tau = 0$ . It then contracts through the ergosphere and the two horizons and eventually falls into  $r = 0$ . If  $J \neq 0$  it may however still be possible to continue this solution into another universe as in the case **(i)**.

**(iii):**  $J^2 = 4E^2$  is the limiting case where the maximal string radius equals the static limit. The potential is exactly zero for  $r = 0$  so also in this case the string contracts through the two horizons and eventually falls into  $r = 0$ .

The exact and complete mathematical solution can be obtained in terms of elementary or elliptic functions, the details can be found in Ref.[8]. In all cases we find only bounded string size solutions and no multi-string solutions. See also Table II.

## 5 More Spacetimes

We will now compare the circular strings in the  $2 + 1$  dimensional BH-ADS spacetime and in the equatorial plane of ordinary  $3 + 1$  dimensional black holes. In the most general case it is natural to compare the spacetime metric, Eq. (4.1), with the ordinary  $3 + 1$  dimensional Kerr anti de Sitter spacetime with metric components:

$$g_{tt} = \frac{a^2 \Delta_\theta \sin^2 \theta - \Delta_r}{\rho^2}, \quad g_{rr} = \frac{\rho^2}{\Delta_r}, \quad g_{t\phi} = (\Delta_r - (r^2 + a^2) \Delta_\theta) \frac{a \sin^2 \theta}{\Delta_o \rho^2},$$

$$g_{\phi\phi} = \left( \Delta_\theta (r^2 + a^2)^2 - a^2 \Delta_r \sin^2 \theta \right) \frac{\sin^2 \theta}{\Delta_o^2 \rho^2}, \quad g_{\theta\theta} = \frac{\rho^2}{\Delta_\theta}, \quad (5.1)$$

where we have introduced the notation:

$$\Delta_r = (1 - \frac{1}{3} \Lambda r^2)(r^2 + a^2) - 2Mr, \quad \Delta_\theta = 1 + \frac{1}{3} \Lambda a^2 \cos^2 \theta,$$

$$\Delta_o = 1 + \frac{1}{3} \Lambda a^2, \quad \rho^2 = r^2 + a^2 \cos^2 \theta. \quad (5.2)$$

Here the mass is represented by  $M$  while  $a$  is the specific angular momentum, and a positive  $\Lambda$  corresponds to de Sitter while a negative  $\Lambda$  corresponds to anti de Sitter spacetime. In the equatorial plane ( $\theta = \pi/2$ ) the metric, Eq. (5.1), is in the general form of Eq. (2.1) so that we can use the analysis of Section 2. In the most general case the potential is given by:

$$\begin{aligned}
V(r) = & -\frac{\Lambda}{3\Delta_o}r^4 + \frac{1 - 2\Lambda a^2/3}{\Delta_o}r^2 - \frac{2M(1 + 2\Lambda a^2/3)}{\Delta_o^2}r \\
& + \frac{2a^2 - \Lambda a^2/3 - E^2\Delta_o^2}{\Delta_o} - \frac{4M\Lambda a^4}{3\Delta_o^2} \frac{1}{r} \\
& + \frac{a^2(\Delta_o(a^2 - E^2\Delta_o^2) - 4M^2)}{\Delta_o^2} \frac{1}{r^2} + \frac{2Ma^2(a^2 - E^2\Delta_o^2)}{\Delta_o^2} \frac{1}{r^3} \quad (5.3)
\end{aligned}$$

i.e. the potential covers seven powers in  $r$ . The general solution will therefore involve higher genus elliptic functions. It is furthermore very complicated to deduce the physical properties of the circular strings from the shape of the potential (the zeros etc.) since the invariant string size defined in Eq. (2.14) is non-trivially connected to  $r$ :

$$S(\tau) = \sqrt{\frac{r^2 + a^2}{1 + \Lambda a^2/3} + \frac{2M}{r} \frac{a^2}{(1 + \Lambda a^2/3)^2}}. \quad (5.4)$$

We have exactly solved the string dynamics in a number of spacetimes of the form, Eq. (5.1) [8]. In the cases of Minkowski space, anti de Sitter space, Schwarzschild and Schwarzschild anti de Sitter space, Fig.7., there are only bounded string size solutions and no multi-string solutions. In Schwarzschild de Sitter space, on the other hand, the dynamics outside the Schwarzschild horizon is similar to the dynamics in "pure" de Sitter space, so we find the complicated spectrum of oscillating, expanding and contracting strings and the multi-string solutions, see Table II.

## 6 Perturbations Around the String Center of Mass

To obtain more insight about the string propagation in all these curved spacetimes we solved the string equations of motion and constraints by considering

perturbations around the exact string center of mass solution, refining the approach originally developed by de Vega and Sánchez [5]. In an arbitrary curved spacetime of dimension  $D$ , the string equations of motion and constraints, in the conformal gauge, take the form:

$$\ddot{x}^\mu - x''^\mu + \Gamma_{\rho\sigma}^\mu(\dot{x}^\rho \dot{x}^\sigma - x'^\rho x'^\sigma) = 0, \quad (6.1)$$

$$g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = g_{\mu\nu} (\dot{x}^\mu \dot{x}^\nu + x'^\mu x'^\nu) = 0, \quad (6.2)$$

for  $\mu = 0, 1, \dots, (D-1)$  and prime and dot represent derivative with respect to  $\sigma$  and  $\tau$ , respectively. Consider first the equations of motion, Eq. (6.1). A particular solution is provided by the string center of mass  $q^\mu(\tau)$ :

$$\ddot{q}^\mu + \Gamma_{\rho\sigma}^\mu \dot{q}^\rho \dot{q}^\sigma = 0. \quad (6.3)$$

Then a perturbative series around this solution is developed:

$$x^\mu(\tau, \sigma) = q^\mu(\tau) + \eta^\mu(\tau, \sigma) + \xi^\mu(\tau, \sigma) + \dots \quad (6.4)$$

After insertion of Eq. (6.4) in Eq. (6.1), the equations of motion are to be solved order by order in the expansion.

To zeroth order we just get Eq. (6.3). To first order we find:

$$\ddot{\eta}^\mu + \Gamma_{\rho\sigma,\lambda}^\mu \dot{q}^\rho \dot{q}^\sigma \eta^\lambda + 2\Gamma_{\rho\sigma}^\mu \dot{q}^\rho \dot{\eta}^\sigma - \eta''^\mu = 0. \quad (6.5)$$

The first three terms can be written in covariant form [6], c.f. the ordinary geodesic deviation equation:

$$\dot{q}^\lambda \nabla_\lambda (\dot{q}^\delta \nabla_\delta \eta^\mu) - R_{\epsilon\delta\lambda}^\mu \dot{q}^\epsilon \dot{q}^\delta \eta^\lambda - \eta''^\mu = 0. \quad (6.6)$$

However, we can go one step further. For a massive string, corresponding to the string center of mass fulfilling (in units where  $\alpha' = 1$ ):

$$g_{\mu\nu}(q) \dot{q}^\mu \dot{q}^\nu = -m^2, \quad (6.7)$$

there are  $D-1$  physical polarizations of string perturbations around the geodesic  $q^\mu(\tau)$ . We therefore introduce  $D-1$  normal vectors  $n_R^\mu$ ,  $R = 1, 2, \dots, (D-1)$ :

$$g_{\mu\nu} n_R^\mu \dot{q}^\nu = 0, \quad g_{\mu\nu} n_R^\mu n_S^\nu = \delta_{RS} \quad (6.8)$$



and consider only first order perturbations in the form:

$$\eta^\mu = \delta x^R n_R^\mu, \quad (6.9)$$

where  $\delta x^R$  are the comoving perturbations, i.e. the perturbations as seen by an observer travelling with the center of mass of the string. The normal vectors are not uniquely defined by Eqs. (6.8). In fact, there is a gauge invariance originating from the freedom to make local rotations of the  $(D-1)$ -bein spanned by the normal vectors. For our purposes it is convenient to fix the gauge taking the normal vectors to be covariantly constant:

$$\dot{q}^\mu \nabla_\mu n_R^\nu = 0. \quad (6.10)$$

This is achieved by choosing the basis  $(q^\mu, n_R^\mu)$  obeying the conditions given by Eqs. (6.8) at a given point, and defining it along the geodesic by means of parallel transport. Another useful formula is the completeness relation that takes the form:

$$g^{\mu\nu} = -\frac{1}{m^2} \dot{q}^\mu \dot{q}^\nu + n^{\mu R} n_R^\nu. \quad (6.11)$$

Using Eqs. (6.7)-(6.10) in Eq. (6.6), we find after multiplication by  $g_{\mu\nu} n_S^\nu$  the spacetime invariant formula [8]:

$$(\partial_\tau^2 - \partial_\sigma^2) \delta x_R - R_{\mu\rho\sigma\nu} n_R^\mu n_S^\nu \dot{q}^\rho \dot{q}^\sigma \delta x^S = 0. \quad (6.12)$$

Since the last term depends on  $\sigma$  only through  $\delta x^S$  it is convenient to make a Fourier expansion:

$$\delta x_R(\tau, \sigma) = \sum_n C_{nR}(\tau) e^{-in\sigma} \quad (6.13)$$

Then Eq. (6.12) finally reduces to:

$$\ddot{C}_{nR} + (n^2 \delta_{RS} - R_{\mu\rho\sigma\nu} n_R^\mu n_S^\nu \dot{q}^\rho \dot{q}^\sigma) C_n^S = 0, \quad (6.14)$$

which constitutes a matrix Schrödinger equation with  $\tau$  playing the role of the spatial coordinate. Notice that in the case of constant curvature spacetimes,  $R_{\mu\rho\sigma\nu} \propto (g_{\mu\sigma} g_{\rho\nu} - g_{\mu\nu} g_{\rho\sigma})$ , the "potential" in Eq. (6.14) is obtained directly from the normalization equations (6.7) and (6.8) without any calculations at all.

For the second order perturbations the picture is a little more complicated. Since they couple to the first order perturbations we consider the full set of perturbations  $\xi^\mu$  [5, 6]:

$$\dot{q}^\lambda \nabla_\lambda (\dot{q}^\delta \nabla_\delta \xi^\mu) - R_{\epsilon\delta\lambda}^\mu \dot{q}^\epsilon \dot{q}^\delta \xi^\lambda - \xi^{\mu\mu} = U^\mu, \quad (6.15)$$

where the source  $U^\mu$  is bilinear in the first order perturbations, and explicitly given by:

$$U^\mu = -\Gamma_{\rho\sigma}^\mu (\dot{\eta}^\rho \dot{\eta}^\sigma - \eta'^\rho \eta'^\sigma) - 2\Gamma_{\rho\sigma,\lambda}^\mu \dot{q}^\rho \eta^\lambda \dot{\eta}^\sigma - \frac{1}{2}\Gamma_{\rho\sigma,\lambda\delta}^\mu \dot{q}^\rho \dot{q}^\sigma \eta^\lambda \eta^\delta. \quad (6.16)$$

After solving Eqs. (6.14)-(6.15) for the first and second order perturbations, the constraints, Eq. (6.2), have to be imposed. In world-sheet light cone coordinates ( $\sigma^\pm = \tau \pm \sigma$ ) the constraints take the form:

$$T_{\pm\pm} = g_{\mu\nu} \partial_\pm x^\mu \partial_\pm x^\nu = 0, \quad (6.17)$$

where  $\partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$ . The world-sheet energy-momentum tensor  $T_{\pm\pm}$  is conserved, as can be easily verified using Eq. (6.1), and therefore can be written:

$$T_{--} = \frac{1}{2\pi} \sum_n \tilde{L}_n e^{-in(\sigma-\tau)}, \quad T_{++} = \frac{1}{2\pi} \sum_n L_n e^{-in(\sigma+\tau)}. \quad (6.18)$$

At the classical level under consideration here, the constraints are then simply:

$$L_n = \tilde{L}_n = 0. \quad (6.19)$$

Up to second order in the expansion around the string center of mass we find:

$$\begin{aligned} T_{\pm\pm} = & -\frac{1}{4}m^2 + g_{\mu\nu} \dot{q}^\mu \partial_\pm \eta^\nu + \frac{1}{4}g_{\mu\nu,\rho} \dot{q}^\mu \dot{q}^\nu \eta^\rho \\ & + g_{\mu\nu} \dot{q}^\mu \partial_\pm \xi^\nu + g_{\mu\nu} \partial_\pm \eta^\mu \partial_\pm \eta^\nu + g_{\mu\nu,\rho} \dot{q}^\mu \eta^\rho \partial_\pm \eta^\nu \\ & + \frac{1}{4}g_{\mu\nu,\rho} \dot{q}^\mu \dot{q}^\nu \xi^\rho + \frac{1}{8}g_{\mu\nu,\rho\sigma} \dot{q}^\mu \dot{q}^\nu \eta^\rho \eta^\sigma \end{aligned} \quad (6.20)$$

and the conditions  $L_0 = \tilde{L}_0 = 0$ , Eq. (6.19), then give a formula for the string mass.

In the case of ordinary  $D \geq 4$  de Sitter space, the first order perturbations, Eq. (6.14), are determined by [5]:

$$\ddot{C}_{nR} + (n^2 - m^2 H^2) C_{nR} = 0. \quad (6.21)$$

For  $mH > |n|$  the frequencies become imaginary and classical instabilities develop. After solving the second order perturbation equations, Eqs. (6.15)-(6.16), the mass formula is found [5]:

$$m^2 = 2 \sum_n (2n^2 - m^2 H^2) \sum_R A_{nR} \tilde{A}_{-nR} \quad (6.22)$$

After quantization one finds that real mass states can only be defined up to some maximal mass [5]. This is reminiscent of the classical string instabilities in de Sitter space. In ordinary anti de Sitter space and in the  $2+1$  black hole Ads (which is locally, but not globally, Ads) the first order perturbation equations and mass formula take the form of Eqs. (6.21)-(6.22), but with  $H^2$  replaced by  $-H^2$  (or  $-1/l^2$  depending on notation). It follows that in these cases there are no instabilities neither classically nor quantum mechanically. The perturbation series approach is perfectly well-defined in these cases.

It is interesting to compare the  $2+1$  black hole Ads with ordinary  $D \geq 4$  black hole Ads. In Secs. 4,5 we found that the circular string motion is very similar in these backgrounds (see Table II). This was actually somewhat surprising since the backgrounds are really very different: the ordinary black holes have a strong curvature singularity at  $r = 0$ , the  $2+1$  black hole Ads has not. The circular strings are however very special configurations and that could be the reason why we did not really see any qualitative differences. For generic strings we would expect some differences, however, and that is indeed what we find when using the string perturbation series approach. In the ordinary black hole Ads spacetime the first order perturbation equations become [8]:

$$\ddot{C}_{n\perp} + (n^2 + m^2 H^2 + \frac{Mm^2}{r^3}) C_{n\perp} = 0, \quad (6.23)$$

$$\ddot{C}_{n\parallel} + (n^2 + m^2 H^2 - \frac{2Mm^2}{r^3}) C_{n\parallel} = 0, \quad (6.24)$$

for the transverse and longitudinal perturbations, respectively. For the transverse perturbations, Eq. (6.23), the bracket is always positive, thus the frequencies are real and no instabilities arise, not even for  $r \rightarrow 0$ . For the

longitudinal perturbations, on the other hand, the bracket can be negative. In that case imaginary frequencies develop. The ( $|n|=1$ )-instability sets in at:

$$r_{\text{inst.}} = \left( \frac{2Mm^2}{1 + m^2 H^2} \right)^{1/3} \quad (6.25)$$

The higher modes develop instabilities for smaller  $r$ , i.e. closer to the singularity. Similar results are obtained in the black string background [8], see Table III.

## 7 Conclusion

We have studied the string propagation in de Sitter and black hole backgrounds. The main part of the talk concerned the dynamics of circular strings in de Sitter space, with special interest in the physical interpretation of the results of the mathematical analysis. We then compared with results obtained in various black hole backgrounds ( $2+1$  BH-ADS, Schwarzschild-anti de Sitter, Schwarzschild-de Sitter). Finally, more insight about the strings in curved spacetimes was obtained using the string perturbation series approach. The main results and conclusions are summarized in tables I,II,III.

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